THE STANDARD EQUATION FOR IMPACT FORCE

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1. A simple rope model

The spring model is an idealized model for the tension developed in a weighted rope. It is a linear model that is, at worst, still the correct first approximation to actual rope behavior. The spring model assumes that the tension T in the rope is proportional to the relative stretch (Hooke's Law). So if we have a piece of rope of length L and it stretches until its length is $L + y$, then the relative stretch is y/L and the tension in the rope is given by

$$T = k\frac{y}{L}. \tag{1}$$

where k is a constant of proportionality, sometimes called the *rope modulus.* (Unfortunately, the term "modulus" is also used for other related constants.) Since $T = k$ when $y = L$, we can think of k as describing how much force it would take to stretch a piece of rope to double its original length, assuming that the material is capable of sustaining such elongations while still exhibiting the linear behavior of the idealized spring. The units of k are tension force per unit relative stretch, for example, kilonewtons per unit relative stretch.

For a fixed piece of rope of length L, it is perhaps clearer to collect constants and write

$$T = \frac{k}{L}y. \tag{2}$$

This says if you stretch a piece of length L and plot the tension T as a function of the elongation y, the graph will be a straight line (of slope k/L) through the origin. When such graphs are constructed from experimental data with real ropes, a curve is obtained which is close to being a straight line. The most deviation occurs at very small and very large y-values, with a more nearly straight profile in between.

Stretching a rope takes work. Suppose a piece of rope of length L is under no tension. Then the work involved in stretching the rope from length L to length $L + s$ is

$$W = \int_0^s T\,dy = \frac{k}{L}\int_0^s y\,dy = \frac{k}{2L}y^2\Big|_0^s = \frac{k}{2L}s^2. \tag{3}$$

2. The conservation of energy approach to peak load

Suppose that a leader with a length of rope L between him and the belayer takes a fall of length H (so is at distance $H/2$ above the highest pro). The rope, statically held by the belayer, catches the fall, stretching an amount s. This means that the total length of the fall is $H + s$ and the potential energy associated with the fall is $mg(H + s)$, where m is the mass of the falling climber and g is the acceleration due to gravity. Conservation of energy

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requires that the loss of potential energy be accounted for by the work done in stretching the rope. Consequently, it must be that

$$mg(H+s) = \frac{k}{2L}s^2. \tag{4}$$

This is really all there is to the analysis. The rest consists of putting this equation in a convenient form for getting answers. The equation is a quadratic equation whose standard form is

$$\frac{k}{2L}s^2 - (mg)s - mgH = 0. \tag{5}$$

Solving for s (the maximum elongation of the rope in stopping the fall) via the quadratic formula,

$$s = \frac{mg \pm \sqrt{(mg)^2 - 4\left(\frac{k}{2L}\right)(-mgH)}}{2\left(\frac{k}{2L}\right)} \tag{6}$$

$$= \frac{mg \pm \sqrt{(mg)^2 + 2kmg\left(\frac{H}{L}\right)}}{\frac{k}{L}}. \tag{7}$$

The quantity under the square root sign is bigger than mg, so the use of the minus sign from the $\pm$ symbol would give a negative value for s—an extraneous root. Hence the only possible sign is positive.

Multiply both sides of the equation by k/L and use the fact that $T = (k/L)s$ to get

$$T = mg + \sqrt{(mg)^2 + 2kmg\left(\frac{H}{L}\right)}. \tag{8}$$

T is the tension corresponding to maximum rope elongation, i. e. T is the maximum tension developed in the rope. Note that for a climber of a given weight mg, and a rope with a given constant k, the maximum tension in the rope depends only on the ratio H/L and not, for example, on just the height of the fall H. This derivation is why everyone discusses falls in terms of the *fall factor*, which is defined to be the value of H/L. Since the separate values of H and L don't matter, set $r = H/L$ and write

$$T = mg + \sqrt{(mg)^2 + 2kmgr}. \tag{9}$$

Let w denote the falling climber's weight. Then $w = mg$ and Equation (9) becomes

$$T = w + \sqrt{w^2 + 2krw}. \tag{10}$$

Note that k must be calculated in units whose force component agrees with the units of w and T.

In order to use this formula, one needs to know the rope modulus k. Manufacturers do not publish the value of k, instead they give the UIAA impact force U, which is the tension developed in the rope when an 80 kg mass undergoes a fall with fall factor $F = 1.78$. This data can be substituted into the equation above: $T = U$, $w = mg = 80 \times 0.0098 = .784$ kN, $r = 1.78$ and the result solved for k to get

$$k = \frac{U(U - 1.568)}{2.791}. \tag{11}$$

The tension equation for an 80 kg climber becomes

$$T = 0.784 + \sqrt{0.784^2 + \frac{U(U - 1.568)}{1.396}(0.784)r}, \tag{12}$$

which simplifies to

$$T = 0.784 + \sqrt{0.615 + (0.562)U(U - 1.568)r}, \tag{13}$$

where U is the rope's published impact force in kN and r is the fall factor. The result for T is in kN.

The peak load on the top piece is conventionally calculated to be $(5/3)T$, which accounts for a frictional force over the top carabiner equal to 1/3 of the rope tension T on the leader's side.

There is a consequence of Equation (10) that seems little appreciated in the climbing world. Simply weighting a rope, rather than having it catch a fall, corresponds to a fall factor $r = 0$. For that fall factor, we find that $T = 2w$; the maximum tension in the climbing rope is *double* the climber's weight. What happens when a rope is weighted is that it stretches until the maximum tension is twice the climber's weight, and then recovers to the point that the tension in the rope is just the climber's weight. But the anchor will be momentarily subjected to double the climber's weight.

3. The differential equation approach to peak load

Differential equations can be used to model the motion of a weighted spring that has been either stretched or compressed and then released. The ingredients needed for the equation are Hooke's Law from Section 1 and Newton's law $F = ma$. The derivation can be found in any book on ordinary differential equations. Repeating it here allows us, first, to alter details of the standard treatments so that the solution is adapted to the particular conditions of the falling leader, and second, to suppress everything that is not relevant to the case at hand.

This approach is longer and more complex than the conservation of energy argument. This disadvantage is offset by a significant advantage: the differential equation approach allows for modifications that produce a more accurate model. It is an observation that the leader does not bounce up and down forever when stopped by a rope, in fact there is hardly any noticeable bounce. For those that know what the terms mean, this suggests that the rope behaves like a critically damped spring, and a damping term can be added to Equation (14) to model that effect. We leave the results of such an improved model to the reader who knows how to carry out the modifications needed.

We, however, are modelling the "rope with falling leader" with a spring, anchored at one end and with a mass mg attached to the other end. Let $y(t)$ denote the amount of displacement in the spring at time t, with downward displacements (stretchings) counted as positive and upward displacements (compressions) counted as negative. At any instant t, there will be a gravitational force mg acting down on the spring and a Hooke's Law force $-\frac{k}{L}y$ acting in the opposite direction from the direction of the displacement y.

Thus, the net force F on the mass at the end of the spring will be $F = mg - \frac{k}{L}y$. Using Newton's law $F = ma$, this becomes $ma = mg - \frac{k}{L}y$ or $a = g - \frac{k}{mL}y$. Using the definition of acceleration, this becomes

$$\frac{d^2y}{dt^2} = g - \left(\frac{k}{mL}\right)y, \qquad y(0) = 0, \tag{14}$$

where y is the displacement from the hanging position of the unweighted spring.

In order to compute the maximum tension in the spring, we need to know what the spring's maximum elongation is. We know that at the instant of maximum elongation, the spring has stopped the weight's fall, in other words, we know that the weight's velocity v must be zero at the instant of maximum spring elongation. The differential equation (14) relates elongation y to time t, but in order to use these observations, we need a result that relates elongation y to velocity v.

To get the velocity v into the picture, use the fact that

$$\frac{d^2y}{dt^2} = \frac{dv}{dt}. \tag{15}$$

Since we would like to view the weight's velocity v as a function of spring elongation y, we are interested in a differential equation involving y and $\frac{dv}{dy}$. But the chain rule tells us that

$$\frac{dv}{dt} = \frac{dv}{dy} \cdot \frac{dy}{dt} = \frac{dv}{dy} v. \tag{16}$$

Assembling the results of Equations (14), (15), and (16), we get the equation

$$v\frac{dv}{dy} = g - \frac{k}{mL}y. \tag{17}$$

But we also need an initial condition for v corresponding to $y(0) = 0$ in Equation (14). In the initial condition $y(0) = 0$, the time $t = 0$ corresponds to the instant at which the spring is hanging unweighted. In terms of the falling climber, this would be the instant at which the rope straightens out before it starts slowing down the falling climber. The velocity at this moment is the velocity v_0 of the falling climber at the instant the rope begins the catch, and so the initial-value problem becomes

$$v\frac{dv}{dy} = g - \frac{k}{mL}y, \qquad v(0) = v_0. \tag{18}$$

This differential equation is already set up for separation of variables, i. e. we can just integrate both sides with respect to y to get

$$\frac{1}{2}v^2 + c = gy - \frac{k}{2mL}y^2, \tag{19}$$

with the value $c = -\frac{1}{2}v_0^2$ determined by the initial condition $v = v_0$ when $y = 0$. So

$$\frac{1}{2}(v^2 - v_0^2) = gy - \frac{k}{2mL}y^2, \tag{20}$$

or

$$v^2 - v_0^2 = 2gy - \frac{k}{mL}y^2. \tag{21}$$

We want the elongation y when the falling leader's velocity v is zero, so we want to solve

$$-v_0^2 = 2gy - \frac{k}{mL}y^2 \tag{22}$$

for y. Rewriting this as $\frac{k}{mL}y^2 - 2gy - v_0^2 = 0$ and using the quadratic formula, we get

$$y = \frac{2g \pm \sqrt{4g^2 + 4\left(\frac{k}{mL}\right)v_0^2}}{\frac{2k}{mL}} = \frac{g \pm \sqrt{g^2 + \left(\frac{k}{mL}\right)v_0^2}}{\frac{k}{mL}} = \frac{mg \pm \sqrt{m^2g^2 + m\left(\frac{k}{L}\right)v_0^2}}{\frac{k}{L}}. \tag{23}$$

In the case we are interested in, the spring will be stretched (not compressed), so y must be positive. Since the quantity $\sqrt{m^2g^2 + m\left(\frac{k}{L}\right)v_0^2}$ is bigger than mg, using the $-$ part of the $\pm$ sign would produce a negative y and this cannot be. Hence, the formula relevant to our case is

$$y = \frac{mg + \sqrt{(mg)^2 + m\left(\frac{k}{L}\right)v_0^2}}{\frac{k}{L}}. \tag{24}$$

By Hooke's Law, the tension T in the spring at the moment when the leader's velocity has been reduced to zero is $T = \frac{k}{L}y$, and so we have

$$T = mg + \sqrt{(mg)^2 + m\left(\frac{k}{L}\right)v_0^2}, \tag{25}$$

where v_0 is the falling leader's velocity at the moment the rope starts to arrest the fall.

We are left with the task of finding v_0. The climber falls a distance H; the time this takes satisfies $\frac{1}{2}gt^2 = H$, so $t = \sqrt{\frac{2H}{g}}$ and hence $v_0 = gt = \sqrt{2gH}$. Substitutin this value is substituted for v_0 into Equation (25) gives

$$T = mg + \sqrt{(mg)^2 + m\left(\frac{k}{L}\right)(2gH)} = mg + \sqrt{(mg)^2 + 2kmg\left(\frac{H}{L}\right)}, \tag{26}$$

and this is precisely Equation (8), thereby re-establishing the result obtained from the conservation of energy argument.

4. Acknowledgements

Calculations like these have been published many times over. As best as I can tell, their origin goes back to *Belaying the Leader*, Richard M. Leonard and Arnold Wexler, Sierra Club Bulletin **31** (1946), later reprinted with Leonard as editor in the book *Belaying the Leader: An Omnibus on Climbing Safety*, Sierra Club (1956).

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